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Exact solutions for Ising chains in a random field

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Abstract. We present an exact solution for the one-dimensional Ising model in a random field. The distribution of field strengths is site independent, symmetric and has a three-peak structure. The free energy is obtained for all finite temperatures. The low-temperature behaviour is studied in detail. We find an expansion for the free energy in integer powers of temperature. The ground-state energy and the zero-point entropy are calculated explicitly. The specific heat is linear for low temperatures, in contradiction with mean-field theory. The origin of an additional exponential parameter is discussed.

1. Introduction

In recent years considerable effort has been put into the study of the influence of quenched random fields on the Ising model (for reviews see [1]). Exactly solvable models play an important role in clarifying the behaviour of the random-field Ising model (RFIM). Even in one dimension there exist exact solutions only for a few particular distributions of the random magnetic fields h_i. Derrida and co-workers [2] considered the symmetric binary distribution ($h_i = \pm H_r$). Thermodynamic properties and the two-point correlation function had been evaluated by Grinstein and Mukamel [3] using a field distribution where h_i are either plus or minus infinity or zero with certain probabilities. A more realistic distribution had been analysed by Nieuwenhuizen and Luck [4]. For an exponential distribution of the random field they found an exact solution and calculated the thermodynamic properties at any temperatures. In extending the investigations, the authors presented a calculation of the two-point correlation function [5] which has also been evaluated recently for a one-dimensional lattice gas model in a random potential at zero temperature [6].

In this paper we consider a related distribution of the random fields and analyse the thermodynamic behaviour using a method introduced by one of us [7, 8]. Here, the free energy is expressed by a special function $D(u)$ (see equation (2.6)). On the other hand, this function obeys an integral equation which can be replaced by a five-term recurrence relation with non-random coefficients.

Setting

$$h_i = H_r x_i \qquad H_r > 0, -\infty < x_i < \infty \tag{1.1a}$$

the distribution $\rho(x_i)$ of the dimensionless reduced fields x_i has the following form:

$$\rho(x_i) = (1-p)\delta(x_i) + (p/2)|x_i|e^{-|x_i|}. \tag{1.1b}$$

The strength H_r of the random fields and the dilution probability p are two parameters of the model. The distribution reveals two maxima and a central peak which vanishes in the undiluted case $p = 1$, and mimics the diluted binary distribution $h_i = 0$ (probability $1-p$) and $h_i = \pm H_r$ (probability $p/2$).

Nieuwenhuizen and Luck [4] have solved the problem for a distribution of the type (1.1b), without the prefactor $|x_i|$ in the last term.

The RFIM with the more general distribution

$$\rho(x_i) = (1-p)\delta(x_i) + (|x_i|^{\nu}/\nu!)(p/2)\mathrm{e}^{-|x_i|} \qquad \nu = 0, 1, 2\ldots \tag{1.2}$$

had been studied in the framework of the mean-field approximation only [9]. For $\nu \to \infty$ (1.2) becomes a diluted binary distribution.

The paper is organised as follows. In section 2 we present some general formalism. The solution of the problem is given in section 3. Section 4 is devoted to the analysis of the low-temperature behaviour of the free energy, the entropy and the specific heat. Section 5 contains concluding remarks and some generalisations.

2. Generalities

In this section we present the general formalism needed for the calculation of thermodynamics functions.

The Hamiltonian for the one-dimensional Ising model in a random field with periodic boundary conditions is given by

$$H = -J \sum_{i=1}^{N} \sigma_{i+1}\sigma_i - \sum_{i=1}^{N} h_i\sigma_i \tag{2.1}$$

where the h_i are independent random variables with the probability distribution (1.1). We restrict ourselves to the ferromagnetic case $J > 0$.

A common way to calculate the free energy is to use the transfer matrix method. Defining a general transfer matrix

$$T_i = \begin{pmatrix} a_i & b_i \\ d_i & c_i \end{pmatrix} \tag{2.2}$$

where the elements a, b, c and d will be specified below (they should be positive only). The partition function of a finite chain having N sites is given by

$$Z_N = \operatorname{tr} \prod_{i=1}^{N} T_i = \operatorname{tr}\begin{pmatrix} A_N & B_N \\ D_N & C_N \end{pmatrix}. \tag{2.3}$$

The quenched free energy of the model is equal to [2]

$$\begin{aligned} \beta F &= -\lim_{N\to\infty} \frac{1}{N} \ln Z_N = -\lim_{N\to\infty} \langle \ln(D_{N+1}/D_N)\rangle_h \\ &= -\lim_{N\to\infty} \langle \ln(d_N R_N + c_N)\rangle_h \end{aligned} \tag{2.4}$$

where we have introduced $R_N = A_N/D_N$ (note that $D_{N+1} = d_N A_N + c_N D_N$). The brackets denote the average with respect to the random field distribution.

The R_i obey the recurrence relation

$$R_{i+1}=\frac{a_iR_i+b_i}{d_iR_i+c_i} \tag{2.5}$$

for all sites i.

It can be shown [10] that as i goes to infinity the distribution of the R_i has a well defined limit which is stationary in the sense that it is invariant under the substitution (2.5).

Following the previous paper [4] we introduce a function of a complex variable u:

$$D(u)=\langle\ln(R-u)\rangle_R. \tag{2.6}$$

Here the bracket $\langle\ldots\rangle_R$ denotes the average with respect to the above-mentioned stationary distribution. The free energy F can be expressed as

$$\beta F=-\langle\ln d\rangle_h-\langle D(-c/d)\rangle_h. \tag{2.7}$$

For simplicity we have suppressed the index i.

Rewriting (2.7) with the help of (2.5) and (2.6), one gets

$$\left\langle D\left(\frac{cu-b}{a-du}\right)\right\rangle_h=D(u)-\langle\ln((a/d)-u)\rangle_h-\langle\ln d\rangle_h-\beta F. \tag{2.8}$$

Apart from a further transformation of the function $D(u)$, the aim will be to derive a differential equation for the expression on the left-hand side in (2.8). As the result of the procedure we obtain two independent equations, namely the already mentioned differential equation and (2.8). In this way we are able to eliminate the left-hand side of (2.8) and after that we can calculate the free energy.

For the further steps it is convenient to use the common representation of the transfer matrix

$$T_i=\begin{pmatrix}\exp(\beta J+\beta h_i) & \exp(-\beta J-\beta h_i)\\ \exp(-\beta J+\beta h_i) & \exp(\beta J-\beta h_i)\end{pmatrix}. \tag{2.9}$$

An alternative approach will be given elsewhere [11].

The recurrence relation (2.5), in terms of the elements in (2.9), is

$$R_{i+1}=\frac{\exp(2\beta J+2\beta h_i)\,R_i+1}{\exp(2\beta h_i)R_i+\exp(2\beta J)}. \tag{2.10}$$

To simplify the calculations we perform a variable transformation replacing $D(u)$ and R by $E(y)$ and V, respectively:

$$E(y)=\langle\ln(V-y)\rangle_V$$

and

$$V=\exp(-\beta J)(\exp(2\beta J)-R^{-1}). \tag{2.11}$$

Using (2.8) and (2.9) the function $E(y)$ obeys the following equation:

$$E(y)-\ln y-\beta F=\langle E(\varphi(x,y))\rangle_x \tag{2.12}$$

with

$$\varphi(x,y)=\exp(\beta J)(\exp(2\beta H_r x)+1)-(2/y)\exp(2\beta H_r x)\sinh(2\beta J).$$

The function $\varphi(x, y)$ satisfies the differential equation

$$\frac{\partial\varphi(x,y)}{\partial x}=a(y)\frac{\partial\varphi(x,y)}{\partial y}$$

with

$$a(y)=y\lambda\left(\frac{y\exp(\beta J)}{2\sinh(2\beta J)}-1\right)\qquad \lambda\equiv 2\beta H_r. \tag{2.13}$$

Introducing the abbreviations

$$\cosh\mu=w^{-1}\exp(\beta J)\qquad w=(2\sinh(2\beta J))^{1/2} \tag{2.14}$$

we get

$$\varphi(0,y)=2w\cosh\mu-w^2/y. \tag{2.15}$$

In the non-random case $p=0$ it results from (2.11) that

$$E(y)=\ln(we^{-\mu}-y)$$

and

$$-\beta F_{\rm pure}=\mu+\ln w. \tag{2.16}$$

Hence the random part of the free energy F_r is given by

$$F_r=F-F_{\rm pure}. \tag{2.17}$$

3. Solution

While the previous calculations are independent of the random field distribution we now consider the special distribution defined in (1.1). To this end it is convenient to introduce a new function $I_\nu^\pm(y)$:

$$I_\nu^\pm(y)=\int_0^\infty dx[E(\varphi(x,y))\pm E(\varphi(-x,y))]\frac{x^\nu}{\nu!}e^{-x} \tag{3.1}$$

where in our case only the values $\nu=0, 1$ are needed.

Integrating (3.1) by parts using (2.13) and the differential operator $L=a(y)\partial/\partial y$, we get

$$I_1^+=2E_0+L(I_0^-+I_1^-)\qquad I_1^-=L(I_0^++I_1^+) \tag{3.2}$$

with $E_0=E(\varphi(0,y))$.

Since (2.12) can be rewritten in the case of the distribution (1.1) as

$$E(y)-\ln y-\beta F=(1-p)E_0+(p/2)I_1^+ \tag{3.3}$$

we need only an equation for I_1^+. It can be derived directly from (3.2):

$$(1-L^2)^2I_1^+=2(1+L^2)E_0. \tag{3.4}$$

Eliminating I_1^+ in (3.3) by using (3.4), we obtain

$$(1-L^2)^2[E(y)-\ln y-(1-p)E_0]=\beta F+p(1+L^2)E_0. \tag{3.5}$$

A further transformation replaces y by z through

$$y=\frac{w(1-z)e^{\mu}}{1-ze^{2\mu}} \tag{3.6}$$

and

$$E(y)=G(z)-\ln(1-ze^{2\mu}) \tag{3.7}$$

leads to the relation

$$\begin{aligned}(1-L^2)^2[G(z)-(1-p)G(ze^{-2\mu})]-p(1+L^2)[G(ze^{-2\mu})-\ln(1-z)]\\ =\beta F+p(1-L^2)^2\ln(1-z)\end{aligned} \tag{3.8}$$

with

$$E_0=G(z\,e^{-2\mu})-\ln(1-z) \qquad L(z)=(\lambda/2)(1-z^2)\partial/\partial z. \tag{3.9}$$

Since (3.8) does not completely determine the function $G(z)$ we have to require certain analyticity properties for $G(z)$ as was discussed in the previous paper [4]. In particular it is possible to expand $G(z)$ around $z=0$:

$$G(z)=G(0)-\sum_{k=1}^{\infty}c_kz^k/k. \tag{3.10}$$

Inserting the last equation in (3.8) and comparing coefficients we get a five-term recurrence relation of the form

$$\begin{aligned}-(\lambda^4/16)[(k+3)(k+2)(k+1)C_{k+4}-4(k+1)(k^2+2k+2)C_{k+2}+2k(3k^2+5)C_k\\ -4(k-1)(k^2-2k+2)C_{k-2}+(k-3)(k-2)(k-1)C_{k-4}]\\ +(\lambda^2/2)[(k+1)C_{k+2}-2kC_k+(k-1)C_{k-2}]\\ =\frac{1-\exp(-2k\mu)}{k(1-(1-p)\exp(-2k\mu))}C_k-(\lambda^2/4)\{(k+1)\exp[-2(k+2)\mu]c_{k+2}\\ -2k\exp(-2k\mu)c_k+(k-1)c_{k-2}\exp[-2(k-2)\mu]\}\end{aligned} \tag{3.11}$$

and

$$\frac{4}{p\lambda^2}F_r=3-2C_2-c_2\exp(-4\mu)+\frac{\lambda^2}{2}(1-4C_2+3C_4) \tag{3.12}$$

where the coefficients c_k and C_k are related by

$$c_k=\frac{pC_k}{1-(1-p)\exp(-2k\mu)}. \tag{3.13}$$

Note that the coefficients c_k and C_k go to zero for large k. Together with the boundary conditions $C_0=1$ and $C_k\to 0$ for $k\to\infty$, equations (3.11) and (3.12) determine the free energy completely, but due to the more complicated random field distribution it cannot be expressed by a continued fraction expansion as in [4].

At low temperatures the number of coefficients needed to get a reasonable accuracy, increases with $\mu^{-1}\sim\exp(2\beta J)$. Therefore the low-temperature behaviour will be discussed separately in the next section.

4. Low-temperature behaviour

To evaluate the low-temperature behaviour we proceed in the following manner. On one side the recurrence relation (3.11) is considered for $k\mu \gg 1$, i.e. $k \to \infty$. In this case (3.11) can be converted in a differential equation. On the other side, the $k\mu \ll 1$ behaviour can be studied more easily from (3.10) directly. Both solutions have to reveal the same behaviour in the region $1 \ll k \ll \mu^{-1}$. For large k, (3.11) is replaced by the fourth-order differential equation

$$-C''''(y)+3C''(y)=u(y)C(y)+[u(y)C(y)]'' \tag{4.1}$$

with $y=\lambda^{-1}\ln(2k\mu)$, where $\lambda \equiv 2\beta H_r$, and

$$u(y)=\frac{1-\exp(-\exp\lambda y)}{1-(1-p)\exp(-\exp\lambda y)}.$$

The asymptotic solutions can be written in the form

$$C(y)=-Ay+B+2\exp(\sqrt{3}y)-2\delta\exp(-\sqrt{3}y) \tag{4.2a}$$

for $y\to-\infty$, where a multiplicative factor has been left out. In the limit $y\to\infty$, one has

$$C(y)=(E+Fy)\,\mathrm{e}^{-y} \tag{4.2b}$$

with arbitrary coefficients A, B, E, F and δ. In the region where $k\mu$ is small, one puts $e^{-2\mu}=1$ in (3.8). The change of variable

$$t=\lambda^{-1}\ln[(1+z)(1-z)^{-1}] \tag{4.3}$$

in terms of which the operator L is given by $L(t)=\partial_t=\partial/\partial t$, leads to (after integration)

$$(\partial_t^3-3\partial_t)G=\frac{3\lambda}{2}-\frac{3\lambda}{\exp(\lambda t)+1}+\lambda^3\frac{\exp(2\lambda t)-\exp(\lambda t)}{(\exp(\lambda t)+1)^3}+\beta F_r t/p. \tag{4.4}$$

The solution of (4.4), in terms of the variable z, is

$$G(z)=\tfrac{1}{2}\ln(1-z^2)-(\beta F_r/6p\lambda^2)\ln^2\frac{1+z}{1-z}$$
$$+c_0-(c_1/2)\left[\left(\frac{1+z}{1-z}\right)^{\alpha}+\left(\frac{1-z}{1+z}\right)^{\alpha}\right] \tag{4.5}$$

with $\alpha=\sqrt{3}/\lambda$ and integration coefficients c_0 and c_1. If one expands (4.5) with respect to z as in (3.10) the coefficients C_k can be expressed for large k as

$$C_{2k}=1+(2\beta F_r/3p\lambda^2)[\ln(4k)+\gamma]+c_1[(4k)^{\alpha}/\Gamma(\alpha)+(4k)^{-\alpha}/\Gamma(-\alpha)]+\mathrm{O}(k^{-1}) \tag{4.6}$$

where γ denotes Euler's constant.

Due to the symmetric field distribution only even coefficients occur in (4.6). Comparing (4.6) with (4.2) in terms of k

$$C_{2k}=B-(A/\lambda)[\ln(4kv)]+(4kv)^{\alpha}-\delta(4kv)^{-\alpha} \tag{4.7}$$

we find the random part of the free energy as a function of the ratio B/A, and the value of δ:

$$F_r=\frac{-3pH_r^2}{J+H_rB/A+T\gamma/2} \qquad \delta=\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\exp(-2\sqrt{3}J/H_r). \tag{4.8}$$

The ratio B/A can be determined as a power series in temperature T using the Laplace transformation. In terms of the Laplace transform

$$D(z)\int_{y_0}^{\infty} C(y)\exp(zy)\,\mathrm{d}y \tag{4.9}$$

equation (4.1) is equivalent to

$$\begin{aligned}(-z^4+3z^2)D(z)+\exp(zy_0)\{C'''+zC''+(z^2-3)C'-z(z^2-3)C\}_{y_0}\\ =(1+z^2)\int D(s+z)\Gamma(1-s/\lambda)f(s/\lambda)\frac{\mathrm{d}s}{2\pi\mathrm{i}s}\\ +\exp(zy_0)[-(uC)'+zuC]_{y_0} \qquad 0<\mathrm{Re}\, s<\lambda, 0<\mathrm{Re}(s+z)<1\end{aligned} \tag{4.10}$$

where $f(s)$ is defined as in [4] by

$$f(s)=p\sum_{n\geqslant 0}(1-p)^n(n+1)^s. \tag{4.11}$$

For $T=0$ (4.10) can be solved immediately because in this case (4.2*a*) is valid for all y in the interval $0\leqslant y<\infty$ and (4.2*b*) in the interval $y_0<y\leqslant 0$, respectively. Inserting (4.2) in (4.9) we get

$$\begin{aligned}D_0(z)=A_0\frac{1-(1-zy_0)\exp(zy_0)}{z^2}+B_0\frac{1-\exp(zy_0)}{z}+2\frac{1-\exp[(z+\sqrt{3})y_0]}{\sqrt{3}+z}\\ +2\delta_0\frac{1-\exp[(z-\sqrt{3})y_0]}{\sqrt{3}-z}+\frac{E_0}{1-z}+\frac{F_0}{(1-z)^2}\end{aligned} \tag{4.12}$$

where the subscript zero at the coefficients indicates zero temperature. Using (4.12) together with (4.10) and taking into account that all terms proportional to $\exp(zy_0)$ cancel, we can evaluate the coefficients

$$A_0=3+2\sqrt{3}-\delta_0(3-2\sqrt{3})$$

$$B_0=4+3\sqrt{3}-\delta_0(4-3\sqrt{3})$$

$$E_0=6+3\sqrt{3}-\delta_0(6-3\sqrt{3})$$

$$F_0=3+3\sqrt{3}-\delta_0(3-3\sqrt{3})$$

with

$$\delta_0=\exp(-2\sqrt{3}J/H_r). \tag{4.13}$$

In deriving (4.13) we have to close the integral contour in the right half plane.

The pole structure of $D_0(z)$ has to be the same as those for finite temperatures since both originate from (4.2). Making for $D(z)$ an ansatz in the same form as in (4.12) with unknown coefficients A, B, E and F, and inserting the expression in (4.10), we obtain, after comparison of the coefficients, the ratio B/A in the form

$$B/A=\sigma_0-\frac{\sigma_1+\gamma}{\lambda}-\frac{\sigma_2}{\lambda^2}+\mathrm{O}(\lambda^{-3})$$

with

$$\sigma_0 = \frac{4+3\sqrt{3}-(4-3\sqrt{3})\delta_0}{\sqrt{3}[2+\sqrt{3}+(2-\sqrt{3})\delta_0]} \tag{4.14a}$$

$$\sigma_1 = \left(\frac{2+\sqrt{3}-(2-\sqrt{3})\delta_0}{2+\sqrt{3}+(2-\sqrt{3})\delta_0}\right)^2 f'(0) \tag{4.14b}$$

$$\sigma_2 = \frac{(f''(0)+\pi^2/6)[2+\sqrt{3}-(2-\sqrt{3})\delta_0]}{[2+\sqrt{3}+(2-\sqrt{3})\delta_0]^2}(1-\delta_0) - f'(0)^2 \frac{[2+\sqrt{3}-(2-\sqrt{3})\delta_0]^2}{[2+\sqrt{3}+(2-\sqrt{3})\delta_0]^3}(1+\delta_0) \tag{4.14c}$$

and

$$f^{(k)}(0) = p \sum_{n\geq 0} (1-p)^n [\ln(n+1)]^k. \tag{4.15}$$

From this we get with the help of (4.8) the free energy

$$F = F_0 + F_1 T + F_2 T^2 + \ldots \tag{4.16}$$

with

$$F_0 = U_0 = -J - \frac{3pH_r^2}{J+H_r\sigma_0} \tag{4.17a}$$

$$-F_1 = S_0 = \frac{3pH_r^2\sigma_1}{2(J+H_r\sigma_0)^2} \tag{4.17b}$$

$$-2F_2 = \Gamma_0 = \lim_{T\to 0} C/T = \frac{3pH_r\sigma_2}{2(J+H_r\sigma_0)^2} + \frac{3pH_r^2\sigma_1^2}{2(J+H_r\sigma_0)^3}. \tag{4.17c}$$

Let us remark that the present method can be used to calculate the free energy to an arbitrary order in T [4, 5]. However, the coefficients do not only depend on the function $f^{(k)}$ (4.15). There appears, beginning at the order T^3, a class of new functions g defined by

$$g[P] = \sum_{n=n}^{\infty} \frac{(-1)^n f(n)}{n!P(n)} \tag{4.18}$$

where $P(n)$ are polynomials in n.

Now we want to discuss the physical contents of (4.17) and compare the results with those obtained in [4] and [9]. As a general feature in our model, we find for the ground-state energy U_0, the zero-point entropy S_0 and the specific heat amplitude Γ_0, a dependence on an exponential term $\delta_0 = \exp(-2\sqrt{3}J/H_r)$ via the coefficients σ_i in (4.14). Its occurrence is unexpected because it was not present in case of the distribution (1.2) with $\nu = 0$ solved in [4]. Its origin is discussed below.

The part of the ground-state energy which is due to the randomness ($U_{0r} = U_0 + J$) tends to $-2pH_r$ if $H_r \gg J$. In this strong-field limit every spin which feels a non-zero field is aligned with it and we have $U_0 \simeq -\langle |h_i| \rangle$, which holds for all random field distributions.

The zero-point entropy S_0 is found to be non-zero in any diluted case due to the sequences of n successive vanishing random fields: $h_1 = h_2 = \ldots = h_n = 0$ while the adjacent ones h_0 and h_{n+1} are antiparallel and large enough. This causes a degeneracy because of the $n+1$ possible locations for the domain walls and a corresponding entropy proportional to $\ln(n+1)$. For $H_r \gg J$ the probability of such a random field configuration is just given by $p^2(1-p)^n$, and one finds $S_0 = pf'(0)/2$, in agreement with our results.

In the very dilute case $p \ll 1$ the zero-point entropy and the specific heat amplitude show logarithmic singularities. Replacing (4.15) for $p \to 0$ by [4]

$$f^{(n)}(0) = |\ln p|^n [1 + n\gamma/\ln p + \dots] \tag{4.19}$$

leads to

$$S_0 \sim \frac{3H_r^2}{2(J+\sigma_0 H_r)^2} p|\ln p| \qquad p \to 0 \tag{4.20}$$

$$\Gamma_0 \sim \frac{3H_r^2}{2(J+\sigma_0 H_r)^3} p|\ln p| \qquad p \to 0. \tag{4.21}$$

In case of strong disorder ($p \to 1$) one finds a linear vanishing zero-point entropy

$$S_0 \sim \tfrac{3}{2}\ln 2 \frac{H_r^2}{(J+\sigma_0 H_r)^2}(1-p) \qquad p \to 1 \tag{4.22}$$

but a finite specific heat amplitude

$$\Gamma_0 \sim \frac{\pi^2 H_r}{4(J+\sigma_0 H_r)^2}(1-\delta_0) \qquad p = 1. \tag{4.23}$$

Comparing these results with those obtained in [4], one observes, apart from prefactors, the same asymptotic behaviour given in (4.20)–(4.22) and a similar behaviour for U_0 and S_0 (4.17). The main difference arises for the specific heat amplitude Γ_0 at $p=1$ (4.23). The linear behaviour $\Gamma_0 \sim H_r^{-1}$ in [4] for a large H_r/J ratio is changed into $\Gamma_0 \sim H_r^{-2}$ due to the additional factor $(1-\delta_0)$.

Now we want to discuss the occurrence of the factor δ_0 in more detail. To this end we refer to a paper by Derrida and Hilhorst [12] in which the authors have obtained the singular behaviour of the free energy of a one-dimensional random system in the form $F(\varepsilon) \simeq c\varepsilon^{2\alpha^*}$ for $\varepsilon \to 0$ and with $\varepsilon = \exp(-2J/T)$. It was assumed that the random field distribution fulfils the conditions $\langle h_i/T \rangle > 0$ and $\langle \exp(-2h_i/T) \rangle > 1$. The exponent α^* is given by the positive root of

$$\langle \exp(-2h_i\alpha/T) \rangle = 1. \tag{4.24}$$

Such an equation is already known for random matrix products (see [12] and references therein), the distribution of a random variable $z = 1 + x_1 + x_1x_2 + \dots$ [13] where the x_i are independent distributed random variables, and also for an asymmetric random field distribution discussed in [4].

Applying (4.24) to our distribution leads to a fourth-order equation for α with the formal solutions

$$\alpha_1 = \alpha_2 = 0 \qquad \alpha_3 = -\alpha_4 = \sqrt{3}/\lambda \qquad (\lambda = 2\beta H_r).$$

Here α_1 and α_2 vanish because our distribution of random fields is symmetric. Consequently, for large J we do not find an exponential decay, but an algebraic one [4]. Nevertheless, the roots α_3 and α_4 show up in our calculations in the form

$$\varepsilon^{2|\alpha|} = \exp(-2\sqrt{3}\,J/H_r) = \delta_0$$

which is exactly the exponential term obtained in our solution. (Note that for example σ_1 (4.14b) can be expressed as $\sigma_1 = f'(0)\tanh^2[\ln(2+\sqrt{3})+\sqrt{3}\,J/H_r]$ and both exponents α_3 and α_4 occur.)

We see that in our case not only $\varepsilon^{2\alpha}$ is present but also $\varepsilon^{2\alpha}$ with the other α from the formal solution of (4.24), e.g. in (4.5)-(4.8) where the evaluated exponent α refers to α_3 and α_4, respectively. This can be understood because in our procedure (4.24) is enclosed in an implicit form. For that reason we consider the characteristic equation of (3.24) in the non-random case ($I_1^+ = 2E_0$). It is easy to check that it coincides with the above-mentioned fourth-order equation for α.

In case of a random field distribution (1.2) with $\nu > 1$ there are similar exponential terms which will be discussed elsewhere [11]. Their precise role is still not understood.

5. Conclusions

Here we have presented an exact solution of a one-dimensional Ising model in a symmetric continuously varying random field with a three-peak structure. The paper is an extension of a previous one [4] with a simpler random field distribution which allows a representation of the free energy by a continued fraction expansion. In our case the free energy is determined by the five-term recurrence relations (3.11) and (3.12) which can be solved numerically at finite temperatures. The low-temperature limit can be studied explicitly using a more refined analysis of the corresponding differential equations (3.8) and (4.1). Instead of an exclusive exponential temperature dependence in the pure case we find an integer power-law behaviour for the free energy in terms of T. The zero-point entropy remains finite for every dilute case $0 < p < 1$. The specific heat behaves linearly in temperature whenever disorder is present.

We can compare this linear behaviour of the specific heat with predictions from mean-field theory. Schneider and Pytte [14] have shown that a mean-field theory is defined in replacing our Hamiltonian (2.1) by

$$H_{\mathrm{MF}} = NJM^2 - \sum_i (2JM + h_i)\sigma_i \tag{5.1}$$

where M is the magnetisation per spin. The free energy per spin is given by

$$F_{\mathrm{MF}} = HM^2 - T\langle \ln 2\cosh(2\beta JM + \beta h_i)\rangle \tag{5.2}$$

where brackets denote the average with respect to the distribution of random fields, and the minimum with respect to M has to be taken. For symmetrical distributions in one dimension, it is natural to assume that $M = 0$ gives the optimum. Hence, one is dealing with independent spins in a random field. For the distributions (1.2) the specific heat follows for low temperatures as

$$C_{\mathrm{MF}} = (T/H)^{1+\nu} p(1+\nu)(2+\nu)\int_0^\infty \frac{x^\nu \mathrm{d}x}{\nu!}\ln(1+\mathrm{e}^{-x}). \tag{5.3}$$

For $\nu = 0$ it behaves linearly, in accordance with the calculations in [4]. For the present situation, $\nu = 1$, (5.3) deviates qualitatively from our exact result (4.17c), which shows a linear specific heat. Also for other values of ν we expect a linear behaviour, in contradiction with (5.3).

The singular behaviour of the zero-point entropy and the specific heat amplitude for $p \to 0$ or $p \to 1$, respectively, is comparable to those obtained in [4], with the exception of the specific heat at $p=1$ which shows a quadratic decay with increasing ratio H_r/J. This is caused by an additional term $1-\delta_0$ in (4.13), which shows up in the considered thermodynamic functions. The occurrence of the parameter δ_0 is discussed in connection with a prediction made by Derrida and Hilhorst [12] concerning a distribution-dependent exponent α (4.24). A better understanding of the role of the Derrida-Hilhorst roots in random field Ising chains is desirable.

Let us finally mention that our method can also be applied to study the more general random field distribution (1.2) with $\nu > 1$. Although some relations (see for instance (3.2)) are written for the general distribution, the concrete verification of the low-temperature behaviour and other properties of the free energy are rather difficult, and will be discussed elsewhere [11].

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